Approximate solution of one-dimensional heat diffusion problems via hybrid profiles

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This paper presents hybrid temperature profiles, a combination of an exponential and a polynomial, capable of providing highly accurate but approximate solutions to one-dimensional heat diffusion problems by the use of the heat balance integral method. After establishing their capability in the case of two test problems, namely, an initially isothermal semi-infinite medium subject to either a specified heat flux or temperature at its surface, two typical applications are considered. These show that the hybrid profile-heat balance integral combination is an excellent approximation that may be used as a first step before launching on a numerical solution in cases that do not have available analytical solutions.

Keywords: *approximating profile; heat diffusion; heat balance integral; phase change*

Introduction

Despite the fact that numerical solutions of heat diffusion problems have become the order of the day, approximate methods still retain some fascination. This is evidenced by the appearance of papers dealing with these methods in the heat transfer literature from time to time.[1-13] Of interest to this study is the heat balance integral (HBI) method first introduced by Goodman.[2] The popularity this method enjoys is due to its inherent simplicity and the ease with which fairly complex problems can be analyzed.

In practice, the HBI method involves the approximation of the actual temperature distribution by a suitable profile. The accuracy of the results depends on the chosen profile. Even though an exponential profile has been used in the published literature,[9,11,12] the polynomial profile has been used more often. Whether all the possibilities have been explored and exhausted is uncertain, nor is it certain that the profiles used thus far are the most accurate. These uncertainties exist because an inherent deficiency of the HBI is that there is no rational basis for selecting or effecting an improvement in a chosen profile. This has prompted some to look for modifications of the HBI method itself, to formulate the problems in terms of the so-called modified HBI or the θ-moment scheme[11,12] and the use of a second integral of the heat equation.[14] However, this study retains the HBI and searches for profiles vastly superior to the often used polynomial profiles.

The search has been successful in discovering new and more accurate profiles. A few typical applications demonstrate these profiles' capability.

Preliminaries

The HBI method has been discussed extensively in the literature. Fixing attention on Test Problem 1 (TP1), the case of the temperature transient in an initially isothermal semi-infinite medium at zero temperature subject to a surface heat flux $q_s = q_0 t^{n/2}$ (see Figure 1), the HBI recognizes a penetration depth $\delta(t)$, beyond which the temperature is zero. The governing equation, assuming constant properties,

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{1}$$

is replaced by the HBI

$$-\alpha \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^{\delta} T\,\mathrm{d}x\right] \tag{2}$$

The initial and boundary conditions are specified as

$$T(x, 0) = T(\delta, t) = 0$$
$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = q_0 t^{n/2} \tag{3}$$

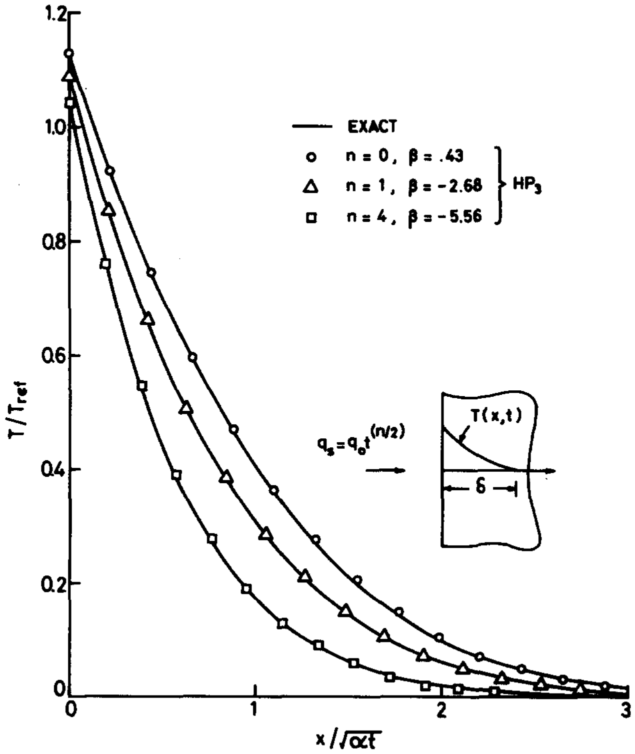

Figure 1 Comparison of hybrid profile with third-degree polynomial (HP_3)—heat balance integral (HBI) temperature profiles with the exact solution for test problem 1 (TP1)

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where the boundary condition $T(\infty, t)=0$ has been applied at $x=\delta$ in the HBI method. In the region $0 \leqslant x \leqslant \delta$, the actual temperature variation is replaced by an approximating profile. Often used polynomial profiles are of the form

$$P_p: \quad T(x,t)=\sum_{m=0}^{p} A_m y^m \qquad y=\frac{x}{\delta} \tag{4}$$

where the coefficients A_m are determined from the boundary or auxiliary conditions as needed and the HBI. In the following, two kinds of profiles are considered: pure exponential profiles (E) and the hybrid profiles (HP_p), a combination of an exponential and a polynomial. These are defined as

$$E: \quad T(x,t)=A_1 \exp(\beta_1 y)+A_2 \exp(\beta_2 y)+A_3 \tag{5}$$

$$HP_p: \quad T(x,t)=A_0 \exp(\beta y)+\sum_{m=1}^{p+1} A_m y^{m-1} \tag{6}$$

When applied to TP1, an evaluation of these profiles may be based on two criteria:

(1) The accuracy with which the surface temperature is predicted
(2) The accuracy with which the temperature profile is predicted

Both these comparisons are made with respect to the well-known exact solution.[15] It is to be expected that the parameter β (or β_1 and β_2) should play a significant role in meeting these two criteria. Apart from the normally satisfied boundary conditions, the smoothness conditions at the edge of the penetration layer, and the HBI, the parameter β makes it possible to satisfy an extra condition that may be chosen arbitrarily. In particular, it is possible that there may be a β for which the surface temperature is predicted exactly. Simple calculation shows, for TP1 with $n=0$, the case of constant heat flux, the following profiles predict a value of $1.1284 q_0 (\alpha t)^{1/2}/k$ that is also predicted by the exact solution:

E: $\beta_1=1.462, \beta_2=0$
HP_1: $\beta=-1.90$
HP_3: $\beta=1.3$
HP_3: $\beta=0.43$

In view of this, the second criterion may be used to choose the best among the four profiles as the suitable profile for this problem. Profile comparisons show HP_3, $\beta=0.43$ is the best profile (Figure 1, $n=0$) in this case. The best profile and the exact profile are in excellent agreement, being indistinguishable from each other, except in a region close to the penetration layer where all profiles underpredict the temperature.

In general (TP1, $n \neq 0$) it can be shown, for profile HP_3, which is chosen as the best candidate, the following hold:

$$\frac{\delta}{(\alpha t)^{1/2}}=\frac{s}{(1+n/2)^{1/2}} =\left\{\frac{\beta \exp(-\beta)-\beta^3/2+\beta^2-\beta}{\beta^{-1}\{1-\exp(-\beta)\}+\beta^3/24-\beta^2/6+\beta/2-1}\right\}^{1/2} \tag{7}$$

$$\frac{T}{T_{ref}}=s\left\{\frac{[\exp\{\beta(y-1)\}-\beta^3 y^3/6+(\beta^3-\beta^2)y^2/2 - (\beta^3/2-\beta^2+\beta)y+(\beta^3/6-\beta^2/2+\beta-1)]}{[\beta \exp(-\beta)-\beta^3/2+\beta^2-\beta]}\right\} \tag{8}$$

Notation

A	Coefficients appearing in the approximating profiles
C	Specific heat of the material of the metal film, J/kg K
Fo	Fourier number in the metal film problem, $\frac{(\alpha t)^{1/2}}{h}$
h	A characteristic length in the metal film problem, $\frac{\alpha_m MC}{k_m}$, m
k	Thermal conductivity, W/m K
L	Latent heat of sublimation in the ablation problem, J/kg
M	Mass per unit area of metal film, kg/m^2
n	Exponent in the surface heat flux variation with time (TP1) or the exponent in the surface temperature variation with time (TP2)
q	Heat flux, W/m^2
r	$\frac{T_s}{T_{ref}}$ in TP1 or $\frac{q_s}{q_{ref}}$ in TP2
s	Reciprocal Fourier number based on penetration depth, $\frac{\delta}{(\alpha t)^{1/2}}$
t	Time, s
T	Temperature, °C
x	Distance measured normal to the surface, m
X	Thickness of material removed in ablation problem, m
y	Nondimensional distance normal to the surface, $\frac{x}{\delta}$
α	Thermal diffusivity, m^2/s
β, β_1, β_2	Constants appearing in the approximating profiles (hybrid)
δ	Depth of penetration, m
Δ	Nondimensional penetration depth in the metal film problem, $\frac{\delta}{h}$
Δ'	Nondimensional penetration depth in the ablation problem, $\frac{\delta - X}{\delta_p}$
λ	Nondimensional ablated layer thickness, $\frac{X}{\delta_p}$
ν	Latent heat to sensible heat ratio in the ablation problem, $\frac{L}{CT_p}$
ρ	Density of medium in the ablation problem, kg/m^3
τ	Nondimensional time in the ablation problem, $\frac{t}{t_p}$

Subscripts

c	Conductive or characteristic
f	Metal film
ℓ	Nondimensional value (after Zien[12])
m	Medium in metal film problem and also 0, 1, 2, ... describing the polynomial profile or the polynomial part of the hybrid profile
n	Nondimensional value (after Zien[12])
p	Degree of the polynomial or phase change value
ref	Suitably chosen reference value (application dependent)
s	Evaluated or specified at the surface
0	Amplitude factor

Table 1 β values satisfying the first criterion, the equality of surface temperature predicted by the HBI with the exact value

n	$\dfrac{T_s}{T_{ref}}$ [b]	β	$\dfrac{s}{(1+n/2)^{1/2}}$
0	1.128	1.300 or 0.430	4.266 or 4.418
1	1.085	−0.279 or −2.680	4.407 or 4.231
2	1.064	−0.249 or −4.095	4.309 or 4.085
3	1.051	−0.233 or −4.940	4.254 or 3.934
4	1.042	−0.229 or −5.560	4.213 or 3.809
5	1.036	−0.226 or −6.015	4.181 or 3.677
6	1.032	−0.223 or −6.400	4.169 or 3.573
7	1.028	−0.220 or −6.700	4.153 or 3.470

[a] Profile is HP_3 applied to TP1.
[b] Exact as well as present approximation. T_{ref} is given by $q_0\sqrt{\alpha t}^{(n+1)/2}/k(1+n/2)$.

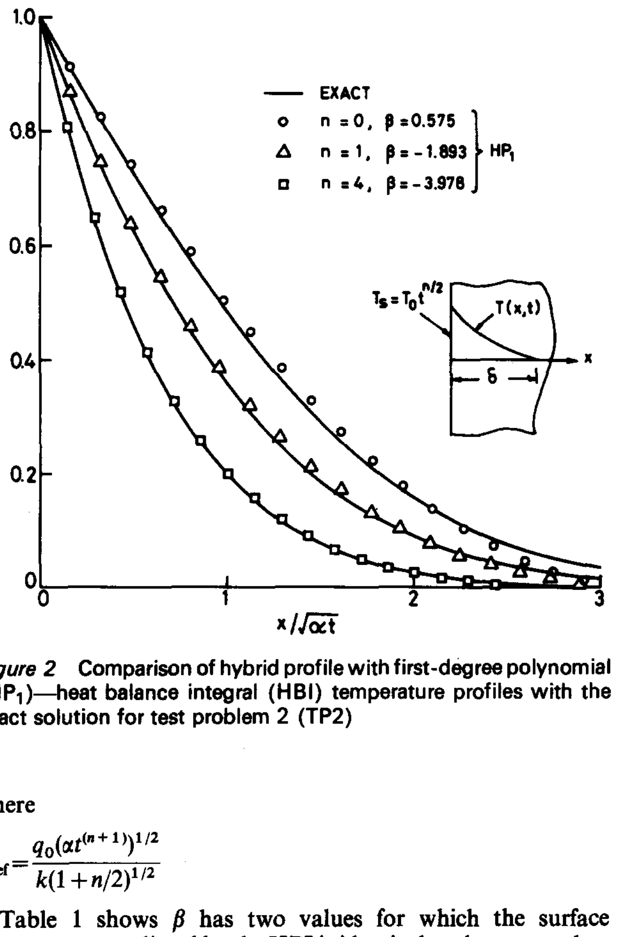

Figure 2 Comparison of hybrid profile with first-degree polynomial (HP_1)—heat balance integral (HBI) temperature profiles with the exact solution for test problem 2 (TP2)

where

$$T_{ref}=\frac{q_0(\alpha t^{(n+1)})^{1/2}}{k(1+n/2)^{1/2}}$$

Table 1 shows β has two values for which the surface temperature predicted by the HBI is identical to the exact value. Invariably the lower value of β yields the better profile. Figure 1 shows, for all n, the HP_3 approximation with this choice of β is excellent and provides solutions largely indistinguishable from the exact profiles.

Now consider Test Problem 2 (TP2), wherein the surface temperature is specified as $T_s=T_0t^{n/2}$, for the case of a semi-infinite medium initially isothermal at zero temperature (see Figure 2). In this case, the first criterion requires the surface heat flux prediction to be identical with that predicted by the exact solution. For $n=0$, the case of a step change in surface temperature, only one profile satisfies this criterion, that is, HP_1, $\beta=0.575$. The nearest rival is P_2, which overpredicts the surface heat flux by 2.3%. In general (TP2, $n\neq 0$), HP_1 satisfies the following:

$$\frac{\delta}{(\alpha t)^{1/2}}=\frac{s}{[(1+n)/2]^{1/2}}=\left\{\frac{\beta[1-\exp(-\beta)]}{[\beta^{-1}-\exp(-\beta)/\beta+\beta/2-1]}\right\}^{1/2} \tag{9}$$

$$\frac{q_s}{q_{ref}}=\frac{\beta[1-\exp(-\beta)]}{s[\exp(-\beta)+\beta-1]} \tag{10}$$

$$\frac{T}{T_s}=\frac{[\exp\{\beta(y-1)\}-\beta y+\beta-1]}{[\exp(-\beta)+\beta-1]} \tag{11}$$

Table 2 shows β has a single value for which the surface heat flux prediction is identical to the exact value. Figure 2 shows, for all n, the corresponding temperature profiles are in excellent agreement with the exact profiles.

Having discovered new hybrid profiles of great accuracy, we consider two interesting applications.

Applications

Transient heating of a semi-infinite medium across a thin partition

This problem models the transient heating of a semi-infinite solid with a surface coating (say, an electro deposition of a film of different metal on a metal sample) or the transient heating of a fluid in a thin-bottomed vessel before the onset of convection.[16] Assuming this film or the vessel bottom is made of a good conductor of heat, we treat the film as a lumped system having a uniform temperature throughout at any time t. The equations governing the problem are (see Figure 3)

$$\text{Metal film:}\quad MC\frac{dT_f}{dt}=q_0-q_c \tag{12}$$

$$\text{Medium:}\quad \alpha_m\frac{\partial^2 T_m}{\partial x^2}=\frac{\partial T_m}{\partial t} \tag{13}$$

with the following initial and boundary conditions:

$$\begin{aligned} &T_f=T_m=0 \quad \text{at } t=0 \text{ for all } x\\ &T_m\to 0 \quad \text{as } x\to\infty \text{ for all } t\\ &T_f=T_m \quad \text{at } x=0 \text{ for all } t\\ &q_c=-k_m\frac{\partial T_m}{\partial x} \quad \text{at } x=0 \end{aligned} \tag{14}$$

Since HP_3 was an excellent profile in the case of TP1, the same profile is chosen in this case also. In this application, it was found that the value of β could be chosen either as 1.3 or 0.43 or any other value in between (like the value $\beta=1$) for predicting the metal film temperature. It can be easily deduced from these equations coupled with HBI and HP_3, $\beta=1$, that the Fourier number $Fo=(\alpha t)^{1/2}/h$ can be obtained as an explicit function of the nondimensional penetration depth $\Delta=\delta/h$ given by

$$Fo=\frac{(3e-8)}{4(e-2)}\left[\Delta^2+\frac{2\Delta}{(e-2)}-\frac{4}{(e-2)^2}\ln\left(1+\frac{\Delta(e-2)}{2}\right)\right]^{1/2} \tag{15}$$

Table 2 β values satisfying first criterion, the equality of surface heat flux predicted by the HBI with the exact value

n	$\dfrac{q_s}{q_{ref}}$ [b]	β	$\dfrac{s}{[(1+n)/2]^{1/2}}$
0	0.798	0.575	3.236
1	0.886	−1.893	3.215
2	0.921	−2.902	3.093
3	0.940	−3.530	2.972
4	0.952	−3.980	2.861
5	0.959	−4.330	2.766
6	0.965	−4.600	2.672
7	0.969	−4.853	2.603

[a] Profile is HP_1 applied to TP2.
[b] Exact as well as present approximation. q_{ref} is given by $kT_0t^{(n-1)/2}[(n+1)/(2\alpha)]^{1/2}$.

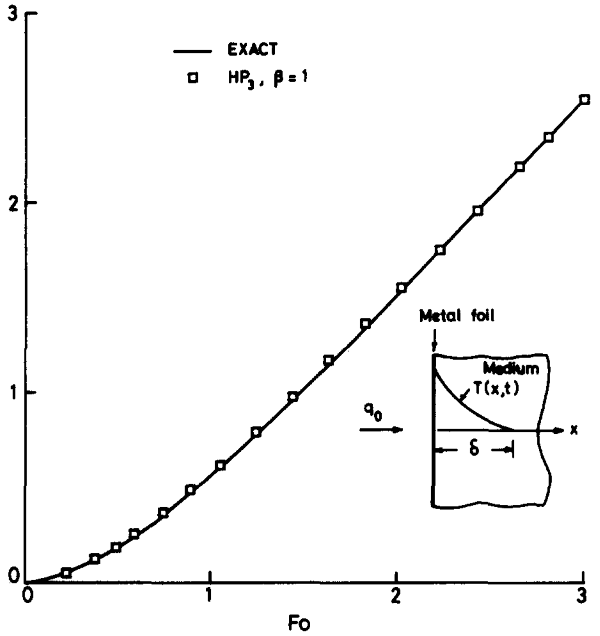

Figure 3 Non-dimensional metal film temperature variation with Fourier number—comparison of the present approximation with the exact solution

The nondimensinal film temperature is then given by

$$\frac{T_f}{T_{ref}}=\frac{(3-e)}{3}\frac{\Delta^2}{[1+\Delta(e-2)/2]} \tag{16}$$

where

$$T_{ref}=\alpha_m\left(\frac{MCq_0}{k_m^2}\right)$$

The exact solution to the problem is obtainable by the application of Laplace transforms.[16] Figure 3 shows the remarkable accuracy with which the present approximation predicts the metal film temperature.

On the physical side, it is interesting to see, after an initial period (Fo ~ 2.5), the film temperature shows a square root dependence on time. This means the transient temperature field in the medium corresponds to that of TP1 with $n=0$ for Fo > 2.5. For short times, however, the heat capacity of the foil becomes important, the conduction into the medium is negligible, and the film temperature increases linearly with time (parabolic with respect to Fo).

Ablation problem

The second application to be treated is the case of ablation under a constant heat flux (see Figure 4). The solution proceeds in two steps. In the first step, the preablation period, the problem corresponds to TP1 with $n=0$ and is valid till a time t_p, at which the surface attains the phase change temperature T_p. Use of HP_3, $\beta=0.43$ and HBI leads to the following values for t_p and δ_p, the depth of penetration at onset of ablation:

$$t_p=\left(\frac{kT_p}{q_0 r\alpha^{1/2}}\right)^2 \qquad \delta_p=\left(\frac{kT_p}{q_0}\right)\left(\frac{s}{r}\right) \tag{17}$$

where $r=1.1284$ and $s=4.4183$.

For the postablation period, the temperature profile is chosen as

$$T=\theta(y) \qquad y=\frac{x-X}{\delta-X} \tag{18}$$

where $X=0$ at $t=t_p$, and $T=T_p$ at $x=X$ (or $y=0$) for all t. The HBI and the energy balance at the ablation front can be shown to be given by, respectively,

$$r'\frac{d\delta'}{dt}+\frac{dX}{dt}=\frac{s'\alpha}{\delta'} \tag{19}$$

$$\frac{s'}{\delta'}=\frac{q_0}{kT_p}-\left(\frac{\rho L}{kT_p}\right)\frac{dX}{dt} \tag{20}$$

where

$$\delta'=\delta-X, \qquad r'=\frac{\int_0^1\theta(y)\,dy}{T_p} \quad \text{and} \quad s'=-\frac{(d\theta/dy)}{T_p} \quad \text{at } y=0$$

These may be nondimensionalized as

$$\lambda=\frac{X}{\delta_p} \qquad \Delta'=\frac{\delta'}{\delta_p} \qquad \tau=\frac{t}{t_p} \tag{21}$$

and integrated subject to the conditions $\lambda=1$, $\Delta'=1$ at $\tau=1$ to get

$$(\tau-1)=a\left[(\Delta'-1)-b\ln\left(\frac{b-\Delta'}{b-1}\right)\right] \tag{22}$$

$$\lambda=\frac{(\tau-1)/rs-r'(\Delta'-1)}{(1+v)} \tag{23}$$

where

$$v=\frac{L}{CT_p} \qquad a=rsr'v \quad \text{and} \quad b=rs'\left(\frac{1+v}{s}\right)$$

Various profiles θ were selected and comparisons with the exact solution[17] showed the best profile for predicting the ablation layer thickness is HP_3, $\beta=0.43$. This is somewhat surprising in the light of the earlier discussion of TP2, $n=0$, where HP_1 turned out to be the best profile. It appears that, though the postablation period corresponds to a constant temperature boundary condition at the receding ablation front, the profile that does not change its form from the preablation to the postablation period is desirable. Fom Figure 4, it is clear that the present HBI result is superior to the result of Vallerani,[9] who used an exponential profile coupled with the HBI. In plotting this figure, the variables used by Zien[12] have been made use of. They are given by

$$\lambda_n=3.7803(1+v)\frac{\lambda}{\tau} \qquad (\tau-\tau_{lp})=0.8(\tau-1) \tag{24}$$

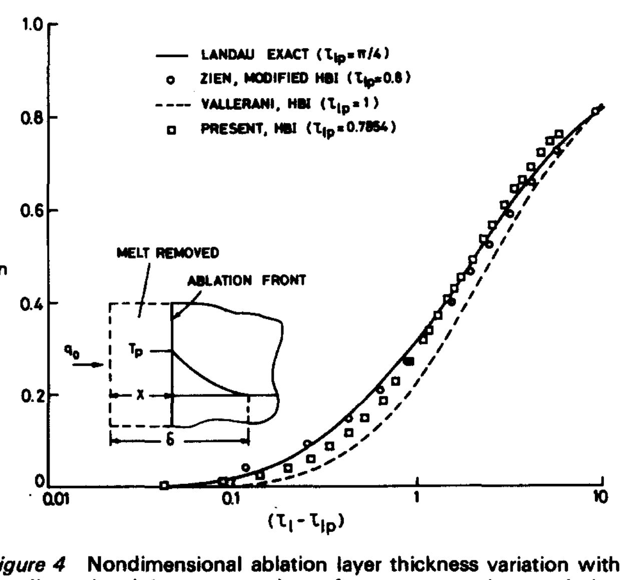

Figure 4 Nondimensional ablation layer thickness variation with nondimensional time—comparison of present approximate solution with other approximate solutions and the exact solution

where

$$\tau_l = \frac{t}{t_c} \qquad \tau_{lp} = \frac{t_p}{t_c} \quad \text{and} \quad t_c = \left(\frac{kT_p}{q_0 \alpha^{1/2}}\right)^2$$

which is a characteristic time. The present result closely follows the exact solution attributable to Landau[17] with the agreement being excellent for short times and relatively long times. The modified HBI method attributable to Zien[12] is in considerable error for short times. On average, the HP_3-HP_3-HBI combination is a very good representation of the solution.

Conclusion

This study has shown the hybrid profiles coupled with the conventional HBI are capable of providing highly accurate solutions to many typical problems in heat diffusion. Here, the applications were limited to cases that had available exact solutions. However, the excellent agreement between the present results and the exact solutions, in each of the cases studied, prompts us to hope that the hybrid profiles should provide accurate solutions when used in problems with more general boundary conditions. In view of this, it is suggested that the hybrid profile HBI combination may be used at least as a first step before undertaking a fully numerical solution in cases not amenable to analytical solution.

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